

Ultimate-state scaling in a shell model for homogeneous turbulent convection

Emily S. C. Ching^{1,2} and T. C. Ko¹¹*Department of Physics, The Chinese University of Hong Kong, Shatin, Hong Kong*²*Institute of Theoretical Physics, The Chinese University of Hong Kong, Shatin, Hong Kong*

(Received 13 March 2008; published 12 September 2008)

An interesting question in turbulent convection is how the heat transport depends on the strength of thermal forcing in the limit of very large thermal forcing. Kraichnan predicted [Phys. Fluids **5**, 1374 (1962)] that for fluids with low Prandtl number (Pr), the heat transport measured by the Nusselt number (Nu) would depend on the strength of thermal forcing measured by the Rayleigh number (Ra) as $Nu \sim Ra^{1/2}$ with logarithmic corrections at very high Ra. According to Kraichnan, the shear boundary layers play a crucial role in giving rise to this so-called ultimate-state scaling. A similar scaling result is predicted by the Grossmann-Lohse theory [J. Fluid Mech. **407**, 27 (2000)], but with the assumption that the ultimate state is a bulk-dominated state in which both the average kinetic and thermal dissipation rates are dominated by contributions from the bulk of the flow with the boundary layers either broken down or playing no role in the heat transport. In this paper, we study the dependence of Nu and the Reynolds number (Re) measuring the root-mean-squared velocity fluctuations on Ra and Pr, for low Pr, using a shell model for homogeneous turbulent convection where buoyancy is acting directly on most of the scales. We find that $Nu \sim Ra^{1/2}Pr^{1/2}$ and $Re \sim Ra^{1/2}Pr^{-1/2}$, which resemble the ultimate-state scaling behavior for fluids with low Pr, and show that the presence of a drag acting on the large scales is crucial in giving rise to such scaling. As a large-scale drag cannot exist by itself in the bulk of turbulent thermal convection, our results indicate that if buoyancy acts on most of the scales in the bulk of turbulent convection at very high Ra, then the ultimate state cannot be bulk dominated.

DOI: [10.1103/PhysRevE.78.036309](https://doi.org/10.1103/PhysRevE.78.036309)

PACS number(s): 47.27.-i, 47.27.te

I. INTRODUCTION

In Rayleigh-Bénard convection, fluid confined in a box is heated from below and cooled on top. When the temperature difference is large enough, convective motion sets in. The flow state is characterized by the geometry of the box and two control parameters: the Rayleigh number (Ra), which measures the strength of the thermal forcing and the Prandtl number (Pr), which is the ratio of the diffusivities of momentum and heat of the fluid. The two parameters are defined by $Ra = \alpha g \Delta L^3 / (\nu \kappa)$, and $Pr = \nu / \kappa$, where Δ is the temperature difference, L is the height of the box, g the acceleration due to gravity, and α , ν , and κ are, respectively the volume expansion coefficient, kinematic viscosity, and thermal diffusivity of the fluid. When Ra is sufficiently high, the convective motion becomes turbulent. Turbulent Rayleigh-Bénard convection has been a system of great research interest (see, for example, Refs. [1,2] for a review). In particular, an interesting question is how the heat transport of the fluid, measured by the Nusselt number (Nu), which is defined as the actual heat transport normalized by that when there were only pure conduction, depends on Ra in the limit of very high Ra. More than 40 years ago, Kraichnan predicted [3] that in this asymptotic limit,

$$Nu \sim Ra^{1/2} (\ln Ra)^{-3/2} Pr^{1/2}, \quad (1)$$

$$Re_0 \sim Ra^{1/2} (\ln Ra)^{-1/2} Pr^{-1/2} \quad (2)$$

for $Pr < 0.15$ and

$$Nu \sim Ra^{1/2} (\ln Ra)^{-3/2} Pr^{-1/4}, \quad (3)$$

$$Re_0 \sim Ra^{1/2} (\ln Ra)^{-1/2} Pr^{-3/4} \quad (4)$$

for $0.15 < Pr \leq 1$. Here $Re_0 = u_0 L / (2\nu)$ is the Reynolds number measuring the root-mean-squared horizontal velocity fluctuations u_0 at midheight $L/2$. Such a scaling behavior of $Nu \sim Ra^{1/2}$ and $Re_0 \sim Ra^{1/2}$ is taken to be a signature of the so-called ultimate state of turbulent convection. This predicted asymptotic increase of Nu as $Ra^{1/2}$ is stronger than the observed dependence of Ra^γ with γ around 0.3 at moderate Ra. According to Kraichnan, the convective eddies, produced by buoyancy in the bulk of the turbulent convective flow, generate turbulent shear boundary layers near the walls and it is the small-scale turbulence present in these boundary layers that dominates and enhances the heat transport. Thus in the picture presented in Kraichnan's work, buoyancy forces are acting directly in the bulk and the shear boundary layers play a crucial role in giving rise to the ultimate-state scaling.

On the other hand, the theory proposed by Grossmann and Lohse [4–6] argued that the kinetic and thermal boundary layers would either break down or do not contribute to the energy and thermal dissipation and thus do not play any role in the heat transport at very high Ra. In this bulk-dominated state, the Grossmann-Lohse (GL) theory predicted that for fluids with low Pr [4]

$$Nu \sim Ra^{1/2} Pr^{1/2}, \quad (5)$$

$$Re_{LSC} \sim Ra^{1/2} Pr^{-1/2}, \quad (6)$$

where $Re_{LSC} = UL / \nu$ is the Reynolds number measuring the mean large-scale circulating flow velocity U near the boundaries. Thus GL predicted the same scaling of $Ra^{1/2}$ for Nu and Re_{LSC} for fluids with low Pr. The Pr-dependence predicted agrees with that of Kraichnan for $Pr < 0.15$, but not for

$0.15 < \text{Pr} \leq 1$. We emphasize that although the predicted asymptotic dependence of Nu and Re_{LSC} or Re_0 for fluids with low Pr are both $\text{Ra}^{1/2}$ in the two theories, the assumed roles of the boundary layers in heat transport in the asymptotic regime are rather different.

The ultimate state of turbulent convection has been elusive [7] in that definitive experimental evidence is lacking. An increase in the Nu - Ra scaling exponent was found around $\text{Ra}=10^{11}$ by Chavanne and co-workers [8,9] in experiments using low-temperature helium gas, and was interpreted as the transition to the ultimate state. However, similar experiments by Niemela *et al.* [10] showed that the measurements of Nu can be well represented by $\text{Nu} \sim \text{Ra}^{0.309}$ for Ra up to around 10^{17} . This puzzling discrepancy in the two experiments remains unresolved. The situation is further complicated by the increasing difficulty to keep the experiments within the Boussinesq approximation at high Ra [11].

GL's work led to the idea of attaining the ultimate-state scaling at moderate Ra by an artificial destruction of the boundary layers [12]. Modeling the bulk of turbulent Rayleigh-Bénard convective flow by three-dimensional homogeneous turbulent convection with periodic boundary conditions [13], Lohse and co-workers [12,14] performed numerical simulations for Pr ranges from 0.1 to 4. They reported results that are consistent with Eqs. (5) and (6) when Re_{LSC} is replaced by the Reynolds number measuring the root-mean-squared velocity fluctuations. It has been found that [13,15] in three-dimensional homogeneous turbulent convection, the Bolgiano length [16], which is an estimate of the length scale above which buoyancy forces are dominant, is comparable to the size of the periodic box. Thus in the numerical model studied by Lohse and co-workers, buoyancy is relevant only at the largest scales. In turbulent Rayleigh-Bénard convection, the Bolgiano length can be expressed as [17]: $l_B = L \text{Nu}^{1/2} (\text{Ra Pr})^{-1/4}$, which, therefore, decreases with Ra at moderate Ra . Thus there is the possibility that buoyancy would become relevant at most scales at very high Ra . The importance of buoyancy at most scales is also in accord with the picture presented in Kraichnan's work [3] as discussed in Sec. I. Hence it is interesting to study the scaling of heat transport using a model for homogeneous turbulent convection in which buoyancy is acting directly on most of the scales.

In this paper, we perform such a study using a shell model for homogeneous turbulent convection in which buoyancy is acting on most of the scales in some parameter range. We investigate the scaling behavior of Nu and the Reynolds number (Re) measuring the root-mean-squared velocity fluctuations for Pr ranges from 0.1 to 2. The paper is organized as follows. In Sec. II, we describe the shell model for homogeneous turbulent convection used in the present study, define Nu , Re , and Ra in the model, and derive two exact results. We present and discuss our results of the dependence of Nu and Re on Ra and Pr in Sec. III. In Sec. IV, we understand why our observed result of the average energy dissipation ϵ is different from that predicted in GL. With the relative simplicity of the shell model, we can derive analytically results for the scaling behavior of Nu and Re . We will illustrate this in Sec. V, and also show that the theoretical results agree well with the numerical observations. Finally, we summarize and conclude in Sec. VI.

II. THE SHELL MODEL FOR HOMOGENEOUS TURBULENT CONVECTION

Homogeneous turbulent convection, which represents the bulk of turbulent Rayleigh-Bénard convection, has been proposed [13] as a three-dimensional convective flow in a box, with periodic boundary conditions, driven by a constant temperature gradient along the vertical direction. In Boussinesq approximation [18], the equations of motion read [12]

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} = -\vec{\nabla} p + \nu \nabla^2 \vec{u} + \alpha g \theta \hat{z}, \quad (7)$$

$$\frac{\partial \theta}{\partial t} + \vec{u} \cdot \vec{\nabla} \theta = \kappa \nabla^2 \theta + \beta u_z, \quad (8)$$

with $\vec{\nabla} \cdot \vec{u} = 0$. Here, \vec{u} is the velocity, p is the pressure divided by the density, $\theta = T - (T_0 - \beta z)$ is the deviation of temperature T from a linear profile of constant temperature gradient of $-\beta$, T_0 is the mean temperature, and \hat{z} is a unit vector in the vertical direction. A dynamical shell model for this system has been proposed by Brandenburg [19]. Shell model is constructed in a discretized Fourier space with $k_n = k_0 h^n$, $n = 0, 1, \dots, N-1$, being the wave number in the n th shell, and h and k_0 are customarily taken to be 2 and 1, respectively. Shell models for homogeneous and isotropic turbulence have been studied extensively and proved to be successful in reproducing the scaling properties observed in experiments [20]. In Brandenburg's model, the velocity and temperature variables u_n and θ_n are real and satisfy the evolution equations,

$$\begin{aligned} \frac{du_n}{dt} = & a k_n (u_{n-1}^2 - h u_n u_{n+1}) + b k_n (u_n u_{n-1} - h u_{n+1}^2) - \nu k_n^2 u_n \\ & + \alpha g \theta_n, \end{aligned} \quad (9)$$

$$\begin{aligned} \frac{d\theta_n}{dt} = & \tilde{a} k_n (u_{n-1} \theta_{n-1} - h u_n \theta_{n+1}) + \tilde{b} k_n (u_n \theta_{n-1} - h u_{n+1} \theta_{n+1}) \\ & - \kappa k_n^2 \theta_n + \beta u_n, \end{aligned} \quad (10)$$

where a , b , \tilde{a} , and \tilde{b} are positive parameters.

For this shell model, it was found [21] that when b/a is larger than some critical value around 2, the effect of buoyancy is greater than the average energy dissipation rate for most shells. Moreover, buoyancy directly affects the statistics of the system such that the scaling behavior of u_n and θ_n is given by the Bolgiano-Obukhov scaling [16,22] ($u_n \sim k_n^{-3/5}$, $\theta_n \sim k_n^{-1/5}$) plus corrections rather than Kolmogorov 1941 scaling [23] ($u_n \sim k_n^{-1/3}$, $\theta_n \sim k_n^{-1/3}$) plus corrections. In other words, buoyancy is directly acting on most of the scales when b/a is sufficiently large. Thus we shall focus on b/a large in the present work. It was reported in earlier studies [19] that the value of b/a controls the direction of energy transfer. For large b/a , there is an inverse energy transfer from small to large scales. The direction of energy transfer can be quantified by the sign of the energy transfer rate. Multiply Eq. (9) by u_n , we obtain

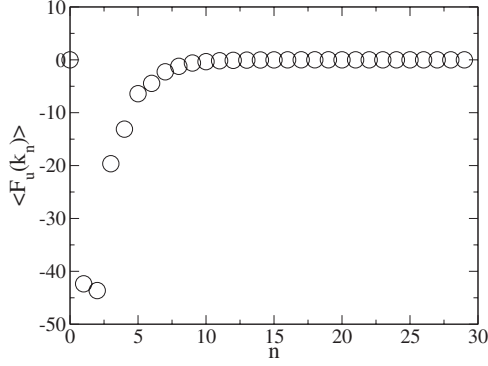


FIG. 1. $F_u(k_n)$ for $a=0.01$, $b=\tilde{a}=\tilde{b}=\alpha g=1$, $\beta=100$, $\nu=\kappa=10^{-8}$.

$$\frac{dE_n}{dt} = F_u(k_n) - F_u(k_{n+1}) - \nu k_n^2 u_n^2 + \alpha g u_n \theta_n, \quad (11)$$

where $E_n = u_n^2/2$ is the energy in the n th shell and

$$F_u(k_n) \equiv k_n (a u_{n-1} + b u_n) u_{n-1} u_n \quad (12)$$

is the rate of energy transfer or energy flux from $(n-1)$ th to n th shell. As shown in Fig. 1, $\langle F_u(k_n) \rangle$ is indeed negative, confirming that, on average, energy is transferred from large n (small scales) to small n (large scales) when b/a is large. As a result, a drag acting on the largest scales has to be added to dampen the growth of energy at large scales so that the system can achieve a statistically stationary state [24–26]. Thus we modify Eq. (9) to

$$\begin{aligned} \frac{du_n}{dt} = & a k_n (u_{n-1}^2 - h u_n u_{n+1}) + b k_n (u_n u_{n-1} - h u_{n+1}^2) - \nu k_n^2 u_n \\ & + \alpha g \theta_n - f u_0 \delta_{n,0}, \end{aligned} \quad (13)$$

where $f u_0 \delta_{n,0}$, with $f > 0$, is a linear drag term acting only on the first shell $n=0$. Also, Eq. (11) becomes

$$\frac{dE_n}{dt} = F_u(k_n) - F_u(k_{n+1}) - \nu k_n^2 u_n^2 + \alpha g u_n \theta_n - f u_0^2 \delta_{n,0}. \quad (14)$$

Next we need to define Ra, Nu, and Re in the shell model noting that Pr is given by the usual definition of ν/κ . The definitions of Ra and Re are straightforward: we only need to replace L by $1/k_0$ such that

$$\text{Ra} = \frac{\alpha g \beta}{k_0^4 \nu \kappa}, \quad (15)$$

$$\text{Re} = \frac{[\sum \langle u_n^2 \rangle]^{1/2}}{\nu k_0}. \quad (16)$$

As for Nu, we recall its definition in turbulent Rayleigh-Bénard convection as

$$\text{Nu} \equiv \frac{\langle u_z (T - T_0) - \kappa \partial T / \partial z \rangle_A}{\kappa \Delta / L} = \frac{\langle u_z \theta \rangle_V}{\kappa \Delta / L} + 1, \quad (17)$$

where $\langle \cdots \rangle_A$ denotes an average over (any) horizontal plane of the convection cell and time, and $\langle \cdots \rangle_V$ denotes an average over the whole volume of the convection cell and time. Thus we define

$$\text{Nu} = \frac{\sum \langle u_n \theta_n \rangle}{\kappa \beta} + 1 \quad (18)$$

accordingly with $\langle \cdots \rangle$ denotes an average over time.

One can derive two exact results in exact analogy to those derived in the case of turbulent Rayleigh-Bénard convection [1]. Multiply Eq. (10) by θ_n , we obtain

$$\frac{dS_n}{dt} = F_\theta(k_n) - F_\theta(k_{n+1}) - \kappa k_n^2 \theta_n^2 + \beta u_n \theta_n, \quad (19)$$

where $S_n \equiv \theta_n^2/2$ is the entropy, in the Bousinesq approximation, in the n th shell and

$$F_\theta(k_n) \equiv k_n (\tilde{a} u_{n-1} + \tilde{b} u_n) \theta_{n-1} \theta_n \quad (20)$$

is the rate of entropy transfer or entropy flux from $(n-1)$ th to n th shell. In the statistically stationary state, summing Eq. (14) and Eq. (19) over n and using Eqs. (18) and (15) give the exact results as

$$\epsilon_{total} = \nu^3 k_0^4 (\text{Nu} - 1) \text{Ra} \text{Pr}^{-2}, \quad (21)$$

$$\chi = \kappa \beta^2 \text{Nu}, \quad (22)$$

where the total energy dissipation rate ϵ_{total} is given by

$$\epsilon_{total} = \epsilon + \epsilon_{drag}. \quad (23)$$

Here ϵ is the average energy dissipation rate given by

$$\epsilon \equiv \nu \sum_n k_n^2 \langle u_n^2 \rangle \quad (24)$$

and ϵ_{drag} is the average rate of energy dissipation due to the large-scale drag,

$$\epsilon_{drag} = f \langle u_0^2 \rangle. \quad (25)$$

The average thermal dissipation rate χ is defined as

$$\chi \equiv \kappa \sum_n k_n^2 \langle \theta_n^2 \rangle + \kappa \beta^2 \quad (26)$$

in accordance with $\chi \equiv \kappa \langle (\nabla T)^2 \rangle_V = \kappa \langle (\nabla \theta)^2 \rangle_V + \kappa \beta^2$ for turbulent Rayleigh-Bénard convection.

III. RESULTS FOR Nu(Ra,Pr) AND Re(Ra,Pr)

We numerically integrate Eqs. (10) and (13) using fourth-order Runge-Kutta method with an initial condition of $u_n = \theta_n = 0$ except for a small perturbation of θ_n in an intermediate value of n . We use $a=0.01$, $b=\tilde{a}=\tilde{b}=\alpha g=1$, $\beta=100$, $f=0.5$, $N=30$, and vary ν and κ to study Nu and Re as a function of Ra and Pr. Five values of Pr, ranging from 0.1 to 2, are studied.

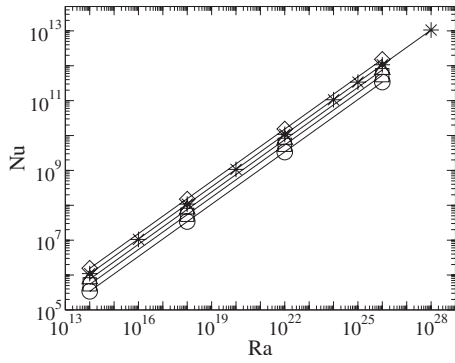


FIG. 2. Dependence of Nu on Ra for for Pr=0.1 (circles), Pr =0.25 (squares), Pr=0.5 (triangles), Pr=1 (stars), and Pr=2 (diamonds). The solid lines are the linear least-square fits of \log_{10} Nu versus \log_{10} Ra.

The dependence of Nu on Ra for each Pr are shown in Fig. 2. It can be seen that for each Pr, Nu scales with Ra: $Nu=A(\text{Pr})Ra^{\gamma_1}$ with $\gamma_1=0.500 \pm 0.001$. The Pr dependence of Nu is found to be $A(\text{Pr})=C_1\text{Pr}^{\gamma_2}$ with $\gamma_2=0.51 \pm 0.01$, as shown in Fig. 3. Thus we have

$$Nu = C_1 Ra^{0.500 \pm 0.001} \text{Pr}^{0.51 \pm 0.01}. \quad (27)$$

Similarly, we study the dependence of Re on Ra for each Pr. As seen in Figs. 4 and 5, $Re=B(\text{Pr})Ra^{\gamma_3}$ and $B(\text{Pr})=C_2\text{Pr}^{\gamma_4}$, where $\gamma_3=0.500 \pm 0.001$ and $\gamma_4=-0.50 \pm 0.01$. Thus

$$Re = C_2 Ra^{0.500 \pm 0.001} \text{Pr}^{-0.50 \pm 0.01}. \quad (28)$$

Hence our results for Nu and Re are consistent with Eqs. (5) and (6), when Re_{LSC} is replaced by Re, and are also consistent with the numerical results [12,14] obtained in the three-dimensional homogeneous turbulent thermal convection in which buoyancy only acts on the largest scales.

Moreover, we note that the Pr dependence of Nu and Re is consistent with GL and not with Kraichnan’s result for $0.15 < \text{Pr} \leq 1$ at very high Ra. The significance of $Nu \sim (Ra \text{Pr})^{1/2}$ and $Re \sim (Ra/\text{Pr})^{1/2}$ is that the heat transport and the root-mean-squared velocity fluctuations are independent of ν and κ . Thus for fluids with low Pr, the scaling $Nu \sim (Ra \text{Pr})^{1/2}$ at very high Ra is in line with a “central dogma in turbulence” [7] that the effects of turbulence become in-

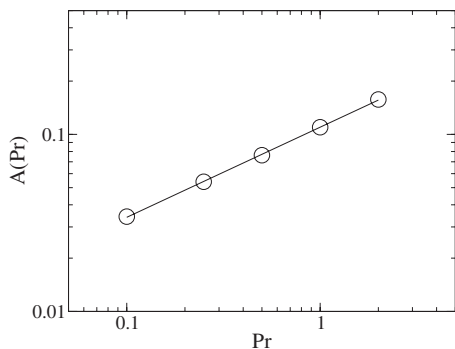


FIG. 3. The prefactor $A(\text{Pr})$ as a function of Pr. The solid line is the linear least-square fit of $\log_{10} A$ versus \log_{10} Pr.

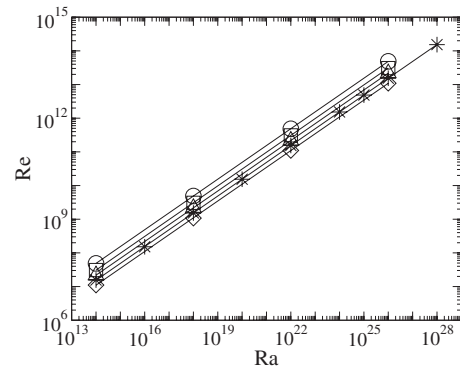


FIG. 4. Dependence of Re on Ra for different values of Pr, with the same symbols as in Fig. 2. The solid lines are the linear least-square fits of \log_{10} Re versus \log_{10} Ra.

dependent of viscosity and thermal diffusivity when Re is sufficiently large. On the other hand, the rigorous upper bound of $Nu \leq 0.167 Ra^{1/2} - 1$ [27], for convection in a layer of fluid with no sidewalls, indicates [28] that the dependence of $Nu \sim (Ra \text{Pr})^{1/2}$ cannot hold for fluids with very large Pr.

In the derivation of Eqs. (5) and (6) in the GL theory, there are two key intermediate results, which are the estimates of ϵ and χ in the bulk-dominated regime for low Pr. In the shell model, $L \sim 1/k_0$, Re_{LSC} becomes Re, and these results [c.f. Eqs. (2.11) and (2.12) of Ref. [4]] translate to

$$\epsilon^{(GL)} \sim \nu^3 k_0^4 Re^3, \quad (29)$$

$$\chi^{(GL)} \sim \kappa \beta^2 Re \text{Pr}. \quad (30)$$

Next we investigate the validity of Eqs. (29) and (30). In Fig. 6, we see that

$$\frac{\epsilon}{\nu^3 k_0^4} = C_3 Re^{2.48 \pm 0.02}. \quad (31)$$

Thus the dependence on Re is approximately $Re^{5/2}$ instead of Re^3 . On the other hand, as can be seen in Fig. 7,

$$\frac{\chi}{\kappa \beta^2} = C_4 (Re \text{Pr})^{1.000 \pm 0.001}, \quad (32)$$

in good agreement with Eq. (30).

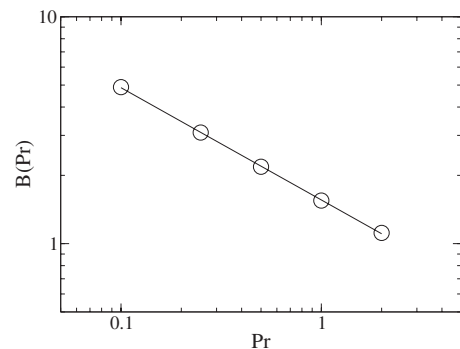


FIG. 5. The prefactor $B(\text{Pr})$ as a function of Pr. The solid line is the linear least-square fit of $\log_{10} B$ versus \log_{10} Pr.

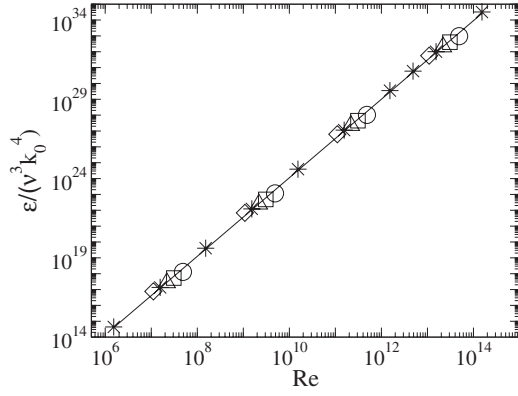


FIG. 6. $\epsilon/(\nu^3 k_0^4)$ as a function of Re for different values of Pr with the same symbols as in Fig. 2. The solid line is the linear least-square fit to all the data points in the log-log plot.

The observed results of Eqs. (31) and (32) together with the exact results Eqs. (21) and (22) imply that ϵ_{total} cannot be dominated by ϵ otherwise we would have $Re \approx (Ra/Pr)^{2/3}$, which is inconsistent with our observed dependence of $Re \sim (Ra/Pr)^{1/2}$. Thus, ϵ_{total} has to be dominated by ϵ_{drag} and that ϵ_{drag} has to scale as Re^3 . We indeed find that $\epsilon \ll \epsilon_{total}$ such that

$$\epsilon_{total} \approx \epsilon_{drag} \quad (33)$$

and as shown in Fig. 8,

$$\frac{\epsilon_{drag}}{\nu^3 k_0^4} = C_5 Re^{3.00 \pm 0.01}, \quad (34)$$

giving consistent results as expected. Hence the observed scaling results of Nu and Re [Eqs. (27) and (28)] depend crucially on the presence of a large-scale drag. Such a damping mechanism at the largest scales cannot exist by itself in the bulk of turbulent thermal convection, but could be a result from the interaction of the boundaries with the buoyancy-generated inverse transfer of energy from small to large scales. Our results, therefore, suggest that when buoyancy is acting directly on most scales, the ultimate state, if it exists, cannot simply be a bulk-dominated flow state. Instead the boundaries must play a crucial role.

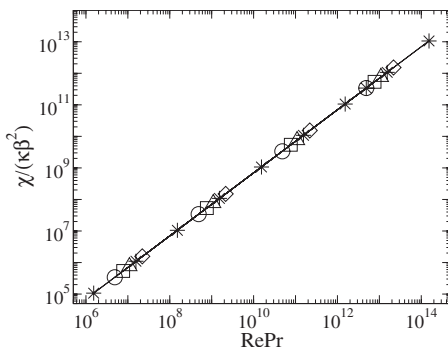


FIG. 7. $\chi/(\kappa\beta^2)$ as a function of $Re Pr$ with the same symbols as in Fig. 2. The solid line is the linear least-square fit to all the data points in the log-log plot.

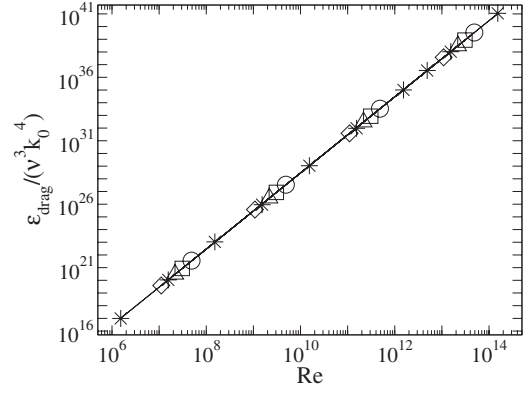


FIG. 8. $\epsilon_{drag}/(\nu^3 k_0^4)$ as a function of Re with the same symbols as in Fig. 2. The solid line is the linear least-square fit to all the datapoints in the log-log plot.

IV. UNDERSTANDING THE DEPENDENCE OF ϵ ON Re

In this section, we discuss how the observed dependence of ϵ on Re approximately as $Re^{5/2}$ can be understood. The average energy dissipation rate in each shell increases with n up to a maximum at the dissipative scale, whose shell is denoted by the shell number n_d , then decreases again. Thus ϵ can be approximated as

$$\epsilon \approx D_1 \nu k_{n_d}^2 \langle u_{n_d}^2 \rangle, \quad (35)$$

where D_1 is a number of order 1. The dissipative wave number k_{n_d} can be estimated as usual as

$$\frac{1}{k_{n_d}} = D_2 \left(\frac{\nu^3}{\epsilon} \right)^{1/4} \quad (36)$$

with D_2 being a number of order 1. Now $\langle u_n^2 \rangle$ has good scaling behavior in k_n [19,21], for $0 \leq n \leq n_d$,

$$\langle u_n^2 \rangle \approx \langle u_0^2 \rangle \left(\frac{k_n}{k_0} \right)^{-2\eta}. \quad (37)$$

Moreover,

$$\sum_n \langle u_n^2 \rangle = D_3 \langle u_0^2 \rangle, \quad (38)$$

where D_3 is a number of order 1. Putting Eqs. (35)–(38) together, we obtain

$$\frac{\epsilon}{\nu^3 k_0^4} \sim Re^{4/(1+\eta)}. \quad (39)$$

As discussed, when buoyancy is acting on most of the scales, the scaling behavior is given by Bolgiano-Obukhov plus corrections, thus $\eta \approx 3/5$. This gives $4/(1+\eta) \approx 5/2$, as observed [see Eq. (31)]. On the other hand, when buoyancy is only acting on the largest scales, u_n obeys Kolmogorov 1941 scaling plus corrections [19]. In this case, $\eta \approx 1/3$, which leads to $4/(1+\eta) \approx 3$, giving the same intermediate result used in the GL theory [see Eq. (29)]. Such a dependence of Re^3 can be easily understood as follows.

When buoyancy is only acting as a driving force at the largest scales, we have the usual energy cascade. From Eq. (11), we have

$$\epsilon \approx \nu k_{n_d}^2 \langle u_{n_d}^2 \rangle \approx \langle F_u(k_{n_d}) \rangle = \langle F_u(k_1) \rangle \approx k_0 \langle u_0^2 \rangle^{3/2} \quad (40)$$

using Eq. (12). Using Eq. (38), Eq. (40) leads immediately to $\epsilon \sim \nu^3 k_0^4 \text{Re}^3$. When buoyancy is acting on most of the scales, there is, however, no longer a cascade of energy but only a cascade of entropy [21]. In particular $\langle F_u(k_{n_d}) \rangle \neq \langle F_u(k_1) \rangle$ and, as a result, ϵ does not scale as Re^3 . In other words, the result Eq. (29) estimated in the GL theory does not hold when buoyancy is directly acting on most of the scales with the dynamics governed by an entropy cascade.

V. ESTIMATES OF χ AND ϵ_{drag} AND THUS $\text{Nu}(\text{Ra}, \text{Pr})$ AND $\text{Re}(\text{Ra}, \text{Pr})$

In the shell model, there is always an entropy cascade with $\langle F_\theta(k_n) \rangle \approx \chi$ being independent of k_n in the intermediate scales. Thus

$$\chi \approx \langle F_\theta(k_1) \rangle \approx k_0 \langle u_0^2 \rangle^{1/2} \langle \theta_0^2 \rangle \quad (41)$$

using Eq. (20). On the other hand, Eq. (19) implies that

$$F_\theta(k_1) \approx \beta \langle u_0 \theta_0 \rangle \approx \beta \langle u_0^2 \rangle^{1/2} \langle \theta_0^2 \rangle^{1/2}. \quad (42)$$

Comparing Eqs. (41) and (42), we obtain

$$k_0 \langle \theta_0^2 \rangle^{1/2} \approx \beta. \quad (43)$$

Putting Eq. (43) into Eq. (41) and using Eq. (38), we obtain

$$\chi \approx \frac{\beta^2}{k_0} \sum_n \langle u_n^2 \rangle^{1/2}, \quad (44)$$

which leads immediately to

$$\chi \approx \kappa \beta^2 \text{Re Pr} \quad (45)$$

as observed [see Eq. (32)].

From Eqs. (25) and (38), we have

$$\epsilon_{drag} \sim f k_0^2 \nu^2 \text{Re}^2 = \nu^3 k_0^4 \frac{f}{\sqrt{\alpha g \beta}} \text{Ra}^{1/2} \text{Pr}^{-1/2} \text{Re}^2. \quad (46)$$

For $\epsilon_{drag} / (\nu^3 k_0^4)$ to depend only on Ra and Pr, we require

$$f = h(\text{Ra}, \text{Pr}) \sqrt{\alpha g \beta} \quad (47)$$

for some function h of Ra and Pr when Ra and Pr are varied. In our numerical calculations, αg , β , and f are kept fixed while ν and κ are varied to obtain different Ra and Pr. This amounts to taking

$$h(\text{Ra}, \text{Pr}) = h_0 \quad (48)$$

for some fixed constant h_0 . Using Eqs. (21), (33), and (46)–(48), we obtain

$$\text{Nu Ra}^{1/2} \approx h_0 \text{Pr}^{3/2} \text{Re}^2. \quad (49)$$

On the other hand, Eqs. (22) and (45) imply

$$\text{Nu} \approx \text{Re Pr}. \quad (50)$$

Solving Eqs. (49) and (50), we obtain

$$\text{Nu} \approx \frac{1}{h_0} \text{Ra}^{1/2} \text{Pr}^{1/2}, \quad (51)$$

$$\text{Re} \approx \frac{1}{h_0} \text{Ra}^{1/2} \text{Pr}^{-1/2}, \quad (52)$$

which are exactly the ultimate-state scaling observed [see Eqs. (27) and (28)]. Substituting Eqs. (47), (48), and (52) into Eq. (46), one obtains

$$\epsilon_{drag} \approx \nu^3 k_0^4 h_0^2 \text{Re}^3, \quad (53)$$

which is in good agreement with our observation [see Eq. (34)].

We have demonstrated that the presence of a drag that acts on the largest scales is crucial for the observation of the ultimate-state scaling of Nu and Re. In our calculations and derivation, we employ a linear drag of the form $f u_0$. An immediate question that arises is whether the scaling laws of Nu and Re depend on the specific mathematical form of the large-scale drag used. To answer this question, we replace $f u_0 \delta_{n,0}$ in Eq. (13) by the general form of a nonlinear drag: $f u_0^{m-1} \delta_{n,0}$, where $m \geq 2$ is an integer, and repeat our analysis to obtain a new estimate of the generalized average energy dissipation rate due to the drag, $\tilde{\epsilon}_{drag}$. Now

$$\tilde{\epsilon}_{drag} = f \langle u_0^m \rangle \approx f \langle u_0^2 \rangle^{m/2}. \quad (54)$$

Using Eq. (38), we obtain

$$\tilde{\epsilon}_{drag} \approx \nu^3 k_0^4 h_0 (\text{Ra Pr}^{-1})^{(3-m)/2} \text{Re}^m \quad (55)$$

where

$$f = h_0 (\sqrt{\alpha g \beta})^{3-m} k_0^{m-2} \quad (56)$$

for some fixed value h_0 as Ra and Pr are varied. Using $\epsilon_{total} \approx \tilde{\epsilon}_{drag}$, Eqs. (21) and (55), we now have

$$\text{Nu Ra}^{(m-1)/2} \approx h_0 \text{Pr}^{(m+1)/2} \text{Re}^m \quad (57)$$

in place of Eq. (49). Together with Eq. (50), which remains the same, we obtain

$$\text{Nu} \approx h_0^{-1/(m-1)} \text{Ra}^{1/2} \text{Pr}^{1/2}, \quad (58)$$

$$\text{Re} \approx h_0^{-1/(m-1)} \text{Ra}^{1/2} \text{Pr}^{-1/2}. \quad (59)$$

In other words, the scaling results of $\text{Nu} \sim (\text{Ra Pr})^{1/2}$ and $\text{Re} \sim (\text{Ra/Pr})^{1/2}$ remain valid for the general large-scale drag of $f u_0^{m-1} \delta_{n,0}$, for $m \geq 2$.

VI. CONCLUSIONS

An interesting question in turbulent thermal convection is how Nu and Re depend on Ra and Pr at very high Ra. Both the theories by Kraichnan [3] and Grossmann and Lohse [4] predicted that in this limit, Nu and Re would scale with $\text{Ra}^{1/2}$ for fluids with low Pr. This kind of scaling behavior is taken to be the signature of the ultimate state of turbulent convection. However, the two theories have rather different assumptions about the role of the boundary layers in heat transport. According to Kraichnan, convective eddies produced in the

bulk generate turbulent shear boundary layers and the turbulence of which enhances heat transport. Thus we interpreted that in the picture presented by Kraichnan, buoyancy is directly acting on all scales in the bulk and that the boundary layers play an important role in heat transport. On the other hand, Grossmann and Lohse argued that in the limit of high Ra, the boundary layers would either break down or not contribute to the energy and thermal dissipation and thus play no role in heat transport. Studying numerically three-dimensional homogeneous turbulent convection in which buoyancy is acting only on the largest scales, Lohse and co-workers [12,14] reported scaling behavior of Nu and Re that is consistent with the ultimate-state scaling.

In the present work, we have studied the scaling behavior of Nu and Re using a shell model of homogeneous turbulent convection in which buoyancy acts on most of the scales. In this model, buoyancy modifies the statistics of the velocity fluctuations such that the statistics are given by Bolgiano-Obukhov plus corrections instead of Kolmogorov 1941 plus corrections [21]. Moreover, there is an average inverse energy transfer from small to large scales such that a large-scale drag has to be present for the system to achieve statistical stationarity. Such a large-scale drag cannot exist by itself in the bulk of turbulent convection but could be a result from the interaction of the inverse energy flow with the boundaries. Thus, when buoyancy is acting directly on most of the scales in the bulk of turbulent convection, the boundary layers would play a crucial role and, as a result, the flow cannot be bulk dominated. In this case, we have found that the dependence of Nu and Re on Ra and Pr is again consistent with the ultimate-state scaling.

Because of the relative simplicity of the shell model, we can understand analytically the scaling behavior of Nu and Re. The two exact results Eqs. (21) and (22) derived for the shell model are in exact analogy to those derived for turbulent Rayleigh-Bénard convection. As is clearly demonstrated by the GL theory, to obtain the Nu and Re scaling, the whole task is to estimate ϵ_{total} and χ . In the shell model, there is always a cascade of entropy so that χ is given by $\kappa\beta^2 \text{Re Pr}$ [see Eq. (45)]. When buoyancy is acting on most of the scales, ϵ_{total} is dominated by ϵ_{drag} . For a linear drag, ϵ_{drag} is

estimated as $\nu^3 k_0^4 (\text{Ra}/\text{Pr})^{1/2} \text{Re}^2$ [see Eq. (53)]. We note that in this case, ϵ is given by $\nu^3 k_0^4 \text{Re}^{5/2}$ [Eq. (39) with $\eta \approx 3/5$ when buoyancy acts on most of the scales] rather than the prediction of $\nu^3 k_0^4 \text{Re}^3$ by the GL theory. Putting these estimates into the two exact results, we obtain the ultimate-state scaling of $\text{Nu} \sim (\text{Ra Pr})^{1/2}$ and $\text{Re} \sim (\text{Ra}/\text{Pr})^{1/2}$. On the other hand, when buoyancy is acting only as a driving force on the largest scales, ϵ_{drag} is negligible and ϵ_{total} given by ϵ as usual. In this case, the statistics of the temperature resemble those of a passive scalar, and ϵ is given by $\nu^3 k_0^4 \text{Re}^3$ [Eq. (39) with $\eta \approx 1/3$ when buoyancy acts only on the largest scales], the usual result obtained in inertia-driven turbulence without buoyancy, which is also the result derived in the GL theory. In this case Eqs. (29) and (30) hold, leading to the ultimate-state scaling as shown in the GL theory.

Hence there are two different physical scenarios that can give rise to the ultimate-state scaling of $\text{Nu} \sim (\text{Ra Pr})^{1/2}$ and $\text{Re} \sim (\text{Ra}/\text{Pr})^{1/2}$ for fluids with low Pr. In the first scenario, which is illustrated in the present work, buoyancy is acting directly on most of the scales of the bulk of turbulent convection and, on average, energy is transferred from small to large scales. An effective damping at the largest scales, which can be provided by the interaction of the inverse energy transfer with the boundaries, is crucial. In the second scenario, buoyancy is acting only as a driving force on the largest scales, temperature in the bulk of the convective flow is behaving similar to a passive scalar statistically, and the boundary layers play no role in heat transport. The first scenario is in accord with the physical picture presented in Kraichnan's work [3] while the second scenario is in accord with that proposed by the GL theory [4]. The next question would be whether or not buoyancy acts directly on most of the scales in the bulk of turbulent Rayleigh-Bénard convection at very high Ra, and the answer of which would help to distinguish which scenario is physically relevant.

ACKNOWLEDGMENTS

The authors thank Roberto Benzi for stimulating discussions and K.R. Sreenivasan for reminding us of Ref. [27]. This work is supported in part by the Hong Kong Research Grants Council (Grant No. CA05/06.SC01).

-
- [1] E. D. Siggia, *Annu. Rev. Fluid Mech.* **26**, 137 (1994).
 - [2] L. P. Kadanoff, *Phys. Today* **54**(8), 34 (2001).
 - [3] R. H. Kraichnan, *Phys. Fluids* **5**, 1374 (1962).
 - [4] S. Grossmann and D. Lohse, *J. Fluid Mech.* **407**, 27 (2000).
 - [5] S. Grossmann and D. Lohse, *Phys. Rev. Lett.* **86**, 3316 (2001).
 - [6] S. Grossmann and D. Lohse, *Phys. Rev. E* **66**, 016305 (2002).
 - [7] J. Sommeria, *Nature (London)* **398**, 294 (1999).
 - [8] X. Chavanne, F. Chillà, B. Castaing, B. Hébral, B. Chabaud, and J. Chaussy, *Phys. Rev. Lett.* **79**, 3648 (1997).
 - [9] X. Chavanne, F. Chillà, B. Chabaud, B. Castaing, and B. Hébral, *Phys. Fluids* **13**, 1300 (2001).
 - [10] J. J. Niemela, L. Skrbek, K. R. Sreenivasan, and R. J. Donnelly, *Nature (London)* **404**, 837 (2000).
 - [11] J. J. Niemela and K. R. Sreenivasan, *J. Fluid Mech.* **481**, 355 (2003).
 - [12] D. Lohse and F. Toschi, *Phys. Rev. Lett.* **90**, 034502 (2003).
 - [13] V. Borue and S. A. Orszag, *J. Sci. Comput.* **12**, 305 (1995).
 - [14] E. Calzavarini, D. Lohse, F. Toschi, and R. Tripiccion, *Phys. Fluids* **17**, 055107 (2005).
 - [15] L. Biferale, E. Calzavarini, F. Toschi, and R. Tripiccion, *Eur. Phys. Lett.* **64**, 461 (2003).
 - [16] R. Bolgiano, *J. Geophys. Res.* **64**, 2226 (1959).
 - [17] See, for example, F. Chilià, S. Ciliberto, C. Innocenti, and E. Pampaloni, *Nuovo Cimento D* **15**, 1229 (1993).
 - [18] See, for example, L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1987).

- [19] A. Brandenburg, Phys. Rev. Lett. **69**, 605 (1992).
- [20] See, for example, L. Biferale, Annu. Rev. Fluid Mech. **35**, 441 (2003).
- [21] E. S. C. Ching and W. C. Cheng, Phys. Rev. E **77**, 015303(R) (2008).
- [22] A. M. Obukhov, Dokl. Akad. Nauk SSSR **125**, 1246 (1959).
- [23] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 9 (1941), reprinted in Proc. R. Soc. London, Ser. A **434**, 9 (1991).
- [24] D. Biskamp, K. Hallatschek, and E. Schwarz, Phys. Rev. E **63**, 045302(R) (2001).
- [25] A. Celani, A. Mazzino, and M. Vergassola, Phys. Fluids **13**, 2133 (2001).
- [26] E. Suzuki and S. Toh, Phys. Rev. E **51**, 5628 (1995).
- [27] C. R. Doering and P. Constantin, Phys. Rev. E **53**, 5957 (1996).
- [28] We thank K. R. Sreenivasan for pointing this out to us.